EM Wave Equation Derivation

Shane O'Brien

September 2023

1 Introduction

The (two-way) wave equation is a second-order linear partial differential equation for the description of waves or standing wave fields (as they occur in classical physics) such as mechanical waves (From Wikipedia). We can re-write Maxwell's Equations, the equations governing electromagnetism, in the form of a classical wave equation, from which we can easily determine the speed at which electromagnetic waves would propagate.

2 Definitions

We can start by defining the Maxwell Equations in free space (where no charges or currents are present):

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, \quad \text{(Gauss's Law)} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{(Faraday's Law)} \\ \nabla \cdot \mathbf{B} &= 0, \quad \text{(Gauss's Law of Magnetism)} \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampère's Law)} \end{aligned} \tag{1}$$

We also need to define the classical wave equation, where v is the propagation speed of the wave and $u = u(x_1, x_2, ..., x_n, t)$ is the funciton that descibes the displacement of the wave in space and time:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u \tag{2}$$

In these derivations, we will also need to define the following curl identity:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \tag{3}$$

3 Derivation from Faraday's Law

We can start by taking the curl of either side of Faraday's Law:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) \tag{4}$$

Then, we can apply the curl identity we stated above and factor the time derivative out of the curl:

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \tag{5}$$

We can recognize the right hand side of this equation as simply the negative of the time derivative of Ampère's Law. Therefore, we can substitute it in:

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{6}$$

We now can see that since we are working in free space with no charge density, $\nabla \cdot \mathbf{E} = 0$ will always be true. Therefore we can further write:

$$\nabla^2 \mathbf{E} = \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \tag{7}$$

Now, as is perhaps obvious, we can write this result in terms of the classical wave equation:

$$\nabla^2 \mathbf{E} = (\mu_0 \epsilon_0)\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{8}$$

Writing the constant term on the other side we get:

$$\boxed{\frac{\partial^2 \mathbf{E}}{\partial t^2} = \left(\frac{1}{\mu_0 \epsilon_0}\right)\nabla^2 \mathbf{E}} \tag{9}$$

Therefore, this gives us the wave equation for E-fields in free-space. The velocity of this wave is:

$$\boxed{v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \cdot 10^8 \mathrm{m/s} = c} \tag{10}$$

Which is, of course, the speed of light in vacuum.

4 Derivation from Ampère's Law

We can use a very similar method to derive the wave equation for B-fields in free-space, this time using Ampère's Law instead. We can start by taking the curl of either side of Ampère's Law:

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \tag{11}$$

Then, we can apply the curl identity we stated above and factor the time derivative out of the curl:

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \tag{12}$$

We can recognize the right hand side of this equation as simply some constants times the time derivative of Faraday's Law. Therefore, we can plug in Faraday's Law to the right hand side of this equation:

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{B}}{\partial t}\right) \tag{13}$$

We also now can note that via Gauss's Law of Magnetism, $\nabla \cdot \mathbf{B} = 0$, the divergence term on the left goes to zero and we are left with:

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{B}}{\partial t}\right) \tag{14}$$

Now, as is perhaps obvious, we can write this result in terms of the classical wave equation:

$$\nabla^2 \mathbf{B} = (\mu_0 \epsilon_0)\frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{15}$$

Writing the constant term on the other side we get:

$$\boxed{\frac{\partial^2 \mathbf{B}}{\partial t^2} = \left(\frac{1}{\mu_0 \epsilon_0}\right)\nabla^2 \mathbf{B}} \tag{16}$$

Therefore, this gives us the wave equation for B-fields in free-space. The velocity of this wave is:

$$\boxed{v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \cdot 10^8 \mathrm{m/s} = c} \tag{17}$$

Which is, once again, the speed of light in vacuum.